Methods for determination and approximation of the domain of attraction

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Abstract

In this paper, an $\mathbb{R}$-analytical function and the sequence of its Taylor polynomials (which are Lyapunov functions different from those of Vanelli & Vidyasagar (1985, Automatica, 21(1):6 9–80)) is presented, in order to determine and approximate the domain of attraction of the exponentially asymptotically stable zero steady state of an autonomous, $\mathbb{R}$-analytical system of differential equations. The analytical function and the sequence of its Taylor polynomials are constructed by recurrence formulae using the coefficients of the power series expansion of f at 0.

Key words: Domain of attraction, Lyapunov function

1 Introduction

Let be the following system of differential equations:

$$\dot{x} = f(x) \tag{1}$$

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where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function of class C^1 on $\mathbb{R}^n$ with $f(0) = 0$ (i.e. $x = 0$ is a steady state of (1)). If the steady state $x = 0$ is asymptotically stable [1], then the set $D_a(0)$ of all initial states x^0 for which the solution $x(t; 0, x^0)$ of the initial value problem:

$$\dot{x} = f(x) \qquad x(0) = x^0 \tag{2}$$

tends to 0 as t tends to ∞, is open and connected and it is called the domain of attraction (domain of asymptotic stability [1]) of 0.

The results of Barbashin [2], Barbashin-Krasovskii [3] and of Zubov ([4], Theorem 19, pp. 52-53, [5]), have probably been the first results concerning the exact determination of $D_a(0)$. In our context, the theorem of Zubov is the following:

Theorem 1 *An invariant and open set S containing the origin and included in the hypersphere $B(r) = \{x \in \mathbb{R}^n : \|x\| < r\}$, $r > 0$, coincides with the domain of attraction $D_a(0)$ if and only if there exist two functions V and ψ with the following properties:*

(1) the function V is defined and continuous on S, and the function ψ is defined and continuous on $\mathbb{R}^n$
(2) $-1 < V(x) < 0$ for any $x \in S \setminus \{0\}$ and $\psi(x) > 0$, for any $x \in \mathbb{R}^n \setminus \{0\}$
(3) $\lim\limits_{x \to 0} V(x) = 0$ and $\lim\limits_{x \to 0} \psi(x) = 0$
(4) for any $\gamma_2 > 0$ small enough, there exist $\gamma_1 > 0$ and $\alpha_1 > 0$ such that $V(x) < -\gamma_1$ and $\psi(x) > \alpha_1$, for $\|x\| \geq \gamma_2$
(5) for any $y \in \partial S$, $\lim\limits_{x \to y} V(x) = -1$
(6) $\frac{d}{dt} V(x(t; 0; x^0)) = \psi(x(t; 0; x^0))[1 + V(x(t; 0; x^0))]$

Remark 2 *At this level of generality, the effective determination of $D_a(0)$ using the functions V and ψ from Zubov's theorem is not possible, because the function V (if ψ is chosen) is constructed by the method of characteristics, using the solutions of system (1). This fact implicitly requests the knowledge of the domain of attraction $D_a(0)$ itself.*

Another interesting result concerning the exact determination of $D_a(0)$, under the hypothesis that the real parts of the eigenvalues of the matrix $\frac{\partial f}{\partial x}(0)$ are negative, is due to Knobloch and Kappel [6]. In our context, Knobloch-Kappel's theorem is the following:

Theorem 3 *If the real parts of the eigenvalues of the matrix $\frac{\partial f}{\partial x}(0)$ are negative, then for any function $\zeta : \mathbb{R}^n \to \mathbb{R}$, with the following properties:*

(1) ζ is of class C^2 on $\mathbb{R}^n$
(2) $\zeta(0) = 0$ and $\zeta(x) > 0$, for any $x \neq 0$

(3) the function ζ has a positive lower limit on every subset of the set $\{x : \|x\| \geq \varepsilon\}$, $\varepsilon > 0$

there exists a unique function V of class C^1 on $D_a(0)$ which satisfies

a. $\langle \nabla V(x), f(x) \rangle = -\zeta(x)$
b. $V(0) = 0$

In addition, V satisfies the following conditions:

c. $V(x) > 0$, *for any* $x \neq 0$
d. $\lim\limits_{x \to y} V(x) = \infty$, *for any* $y \in \partial D_a(0)$ *or for* $\|x\| \to \infty$

Remark 4 *The effective determination of $D_a(0)$ using the functions V and ζ from Knobloch-Kappel's theorem (at this level of generality) is not possible, because the function V (if ζ is chosen) is constructed by the method of characteristics using the solutions of system (1). This fact implicitly requests the knowledge of $D_a(0)$.*

Vanelli and Vidyasagar have established in [7] a result concerning the existence of a maximal Lyapunov function (which characterizes $D_a(0)$), and of a sequence of Lyapunov functions which can be used for approximating the domain of attraction $D_a(0)$. In the context of our paper, the theorem of Vanelli-Vidyasagar is the following:

Theorem 5 *An open set S which contains the origin coincides with the domain of asymptotic stability of the asymptotically stable steady state $x = 0$, if and only if there exists a continuous function $V : S \to \mathbb{R}_+$ and a positive definite function ψ on S with the following properties:*

(1) $V(0) = 0$ *and* $V(x) > 0$, *for any* $x \in S \setminus \{0\}$ *(V is positive definite on S)*
(2) $D_r V(x^0) = \lim\limits_{t \to 0_+} \frac{V(x(t;0,x^0)) - V(x^0)}{t} = -\psi(x^0)$, *for any* $x^0 \in S$
(3) $\lim\limits_{x \to y} V(x) = \infty$, *for any* $y \in \partial S$ *or for* $\|x\| \to \infty$

Remark 6 *The determination of $D_a(0)$ using the functions V and ψ from Vanelli-Vidyasagar's theorem is not possible, for the same reason as in the case of the theorems of Zubov and Knobloch-Kappel.*

Remark 7 *Restraining generality, and considering the case of an $\mathbb{R}$-analytic function f, for which the real parts of the eigenvalues of the matrix $\frac{\partial f}{\partial x}(0)$ are negative, Vanelli and Vidyasagar [7] establish a second theorem which provides a sequence of Lyapunov functions, which are not necessarily maximal, but can be used in order to approximate $D_a(0)$. These Lyapunov functions are of the*

form:

$$V_m(x) = \frac{r_2(x) + r_3(x) + ... + r_m(x)}{1 + q_1(x) + q_2(x) + ... + q_m(x)} \qquad m \in \mathbb{N} \tag{3}$$

where r_i and q_i are i-th degree homogeneous polynomials, constructed using the elements of the matrix $\frac{\partial f}{\partial x}(0)$, of a positively definite matrix G and the nonlinear terms from the development of f. The algorithm of the construction of V_m is relatively complex, but does not suppose knowledge of the solutions of system (1).

Very interesting results concerning the exact determination of the domains of attraction (asymptotic stability domains) have been found by Gruyitch between 1985-1995. These results can be found in [1], chap. 5. In these results, the function V which characterizes the domain of attraction is constructed by the method of characteristics, which uses the solutions of system (1). Some illustrative examples are exceptions because for them V is found in a finite form, for some concrete functions f, but without a precise generally applicable rule.

In the same year as Vanelli and Vidyasagar (1985), Balint [8], proved the following theorem:

Theorem 8 *(see [8] or [9]) If the function f is $\mathbb{R}$-analytic and the real parts of the eigenvalues of the matrix $\frac{\partial f}{\partial x}(0)$ are negative, then the domain of attraction $D_a(0)$ of the asymptotically stable steady state $x = 0$ coincides with the natural domain of analyticity of the $\mathbb{R}$-analytical function V defined by*

$$\langle \nabla V(x), f(x) \rangle = -\|x\|^2 \qquad V(0) = 0 \tag{4}$$

The function V is strictly positive on $D_a(0) \setminus \{0\}$ and $\lim_{x \to y} V(x) = \infty$ for any $y \in \partial D_a(0)$ or for $\|x\| \to \infty$.

Remark 9 *In the case when the matrix $\frac{\partial f}{\partial x}(0)$ is diagonalizable, recurrence formulae have been established in [10] (see also [9]) for the computation of the coefficients of the power series expansion in 0 of the function V defined by (4) (called optimal Lyapunov function in [10]):*

Consider $S : \mathbb{C}^n \to \mathbb{C}^n$ an isomorphism which reduces $\frac{\partial f}{\partial x}(0)$ to the diagonal form $S^{-1}\frac{\partial f}{\partial x}(0)S = diag(\lambda_1, \lambda_2 ... \lambda_n)$. Let be $g = S^{-1} \circ f \circ S$ and $W = V \circ S$. If the expansion of W at 0 is

$$W(z_1, z_2, ..., z_n) = \sum_{m=2}^{\infty} \sum_{|j|=m} B_{j_1 j_2 ... j_n} z_1^{j_1} z_2^{j_2} ... z_n^{j_n} \tag{5}$$

and the expansions at 0 *of the scalar components* g_i *of* g *are*

$$g_i(z_1, z_2, ..., z_n) = \lambda_i z_i + \sum_{m=2}^{\infty} \sum_{|j|=m} b^i_{j_1 j_2 ... j_n} z_1^{j_1} z_2^{j_2} ... z_n^{j_n} \tag{6}$$

then the coefficients $B_{j_1 j_2 ... j_n}$ *of the development (5) are given by the following relations:*

$$B_{j_1 j_2 ... j_n} = \begin{cases} -\frac{1}{2\lambda_{i_0}} \sum\limits_{i=1}^{n} s^2_{i i_0} \ \text{if } |j| = j_{i_0} = 2 \\ \\ -\frac{2}{\lambda_p + \lambda_q} \sum\limits_{i=1}^{n} s_{ip} s_{iq} \ \text{if } |j| = 2 \text{ and } j_p = j_q = 1 \\ \\ -\frac{1}{\sum\limits_{i=1}^{n} j_i \lambda_i} \sum\limits_{p=2}^{|j|-1} \sum\limits_{|k|=p, k_i \leq j_i} \sum\limits_{i=1}^{n} [(j_i - k_i + 1) \\ b^i_{k_1 k_2 ... k_n} B_{j_1 - k_1 ... j_i - k_i + 1 ... j_n - k_n}] \ \text{if } |j| \geq 3 \end{cases} \tag{7}$$

Using these recurrence formulae, the optimal Lyapunov functions V *and the domains of attraction* $D_a(0)$ *for some two-dimensional systems have been found in [10] in a finite form.*

Example 10

$$\begin{cases} \dot{x_1} = -\lambda x_1 + \rho_1 x_1^2 + \rho_2 x_1 x_2 \\ \dot{x_2} = -\lambda x_2 + \rho_1 x_1 x_2 + \rho_2 x_2^2 \end{cases} \qquad \lambda > 0, \rho_1, \rho_1 \in \mathbb{R}^1 \tag{8}$$

The Lyapunov function corresponding to the zero asymptotically stable steady state of this system is

$$V(x_1, x_2) = \frac{x_1^2 + x_2^2}{\lambda} [\frac{\lambda^2}{(\rho_1 x_1 + \rho_2 x_2)^2} \ln \frac{\lambda}{\lambda - (\rho_1 x_1 + \rho_2 x_2)} - \frac{\lambda}{\rho_1 x_1 + \rho_2 x_2}] \tag{9}$$

and the domain of attraction is

$$D_a(0) = \{x \in \mathbb{R}^2 : \rho_1 x_1 + \rho_2 x_2 < \lambda\} \tag{10}$$

Example 11

$$\begin{cases} \dot{x_1} = -\lambda x_1 + \rho x_1^3 + \rho x_1 x_2^2 \\ \dot{x_2} = -\lambda x_2 + \rho x_1^2 x_2 + \rho x_2^3 \end{cases} \qquad \lambda > 0, \rho \in \mathbb{R}^1 \tag{11}$$

The Lyapunov function corresponding to the zero asymptotically stable steady state of this system is

$$V(x_1, x_2) = \frac{1}{2\rho} \ln \frac{\lambda}{\lambda - \rho(x_1^2 + x_2^2)} \tag{12}$$

and the domain of attraction is

$$D_a(0) = \{x \in \mathbb{R}^2 : \lambda - \rho(x_1^2 + x_2^2) > 0\} \tag{13}$$

Therefore, when the function f is $\mathbb{R}$-analytic, the real parts of the eigenvalues of the matrix $\frac{\partial f}{\partial x}(0)$ are negative, and the matrix $\frac{\partial f}{\partial x}(0)$ is diagonalizable, then the optimal Lyapunov function V can be found theoretically by computing the coefficients of its power series expansion at 0, without knowing the solutions of system (1). More precisely, in this way, the "embryo" V_0 (i.e. the sum of the series) of the function V is found theoretically on the domain of convergence D_0 of the power series expansion. A formula for determining the region of convergence $D_0 \subset D_a(0)$ of the series of V can be found in [11] or [9]. If D_0 is a strict part of $D_a(0)$, then the "embryo" V_0 can be prolonged using the algorithm of prolongation of analytic functions:

If D_0 is strictly contained in $D_a(0)$, then there exists a point $x^0 \in \partial D_0$ such that the function V_0 is bounded on a neighborhood of x^0. Let be a point $x_1^0 \in D_0$ close to x^0, and the power series development of V_0 in x_1^0 (the coefficients of this development are determined by the derivatives of V_0 in x_1^0). Using the formula from [11] or [9], the domain of convergence D_1 of the series centered in x_1^0 is obtained, which gives a new part $D_1 \setminus (D_0 \bigcap D_1)$ of the domain of attraction $D_a(0)$. The sum V_1 of the series centered in x_1^0 is a prolongation of the function V_0 to D_1 and coincides with V on D_1. At this step, the part $D_0 \bigcup D_1$ of $D_a(0)$ and the restriction of V to $D_0 \bigcup D_1$ are obtained.

If there exists a point $x^1 \in \partial(D_0 \bigcup D_1)$ such that the function $V|_{D_0 \bigcup D_1}$ is bounded on a neighborhood of x^1, then the domain $D_0 \bigcup D_1$ is strictly included in the domain of attraction $D_a(0)$. In this case, the procedure described above is repeated, in a point x_1^1 close to x^1.

The procedure cannot be continued in the case when it is found that on the boundary of the domain $D_0 \bigcup D_1 \bigcup ... \bigcup D_p$ obtained at step p, there are no points having neighborhoods on which $V|_{D_0 \bigcup D_1 \bigcup ... \bigcup D_p}$ is bounded. We illustrate this process in the following example:

Example 12 Consider the following differential equation:

$$\dot{x} = x(x-1)(x+2) \tag{14}$$

$x = 0$ is an asymptotically stable steady state for this equation. The coefficients of the power series development in 0 of the optimal Lyapunov function are computed using (7): $A_n = \frac{2^{n-1}+(-1)^n}{3n2^{n-1}}$, $n \geq 2$. The domain of convergence $D_0 = (-1, 1)$ of the series is found using the formula:

$$x \in D_0 \qquad \text{iff} \qquad \overline{\lim_n} \sqrt[n]{|A_n x^n|} < 1 \tag{15}$$

The embryo $V_0(x)$ is unbounded in 1 and bounded in -1, as $V_0(-1) = \frac{\ln 2}{3}$. We expand $V_0(x)$ in -0.9 close to -1. The coefficients of the series centered in -0.9 are: $A'_n = \frac{1}{3n}[\frac{1}{(1.9)^n} + \frac{2(-1)^n}{(1.1)^n}]$. The domain of convergence D_1 of the series centered in -0.9 is given by:

$$x \in D_1 \qquad \text{iff} \qquad \overline{\lim_n} \sqrt[n]{|A'_n (x+0.9)^n|} < 1 \tag{16}$$

and it is $D_1 = (-2, 0.2)$. So far, we have obtained the part $D = D_0 \bigcup D_1 = (-2, 1)$ of the domain of attraction $D_a(0)$. As the function V is unbounded at both ends of the interval, we conclude that $D_a(0) = (-2, 1)$.

We have illustrated how this approximation technique described in [8,10,11] works in some particular cases. In more complex cases (for example if the right hand side terms in (1) are just polynomials of second degree), we can only compute effectively the coefficients $A_{j_1 j_2 \ldots j_n}$ of the expansion of V up to a finite degree p. With these coefficients, the Taylor polynomial of degree p corresponding to V:

$$V_0^p(x_1, x_2, ..., x_n) = \sum_{m=2}^{p} \sum_{|j|=m} A_{j_1 j_2 \ldots j_n} x_1^{j_1} x_2^{j_2} ... x_n^{j_n} \tag{17}$$

can be constructed. In the followings, it will be shown how V_0^p can be used in order to approximate $D_a(0)$.

2 Theoretical results

For $r > 0$, we denote by $B(r) = \{x \in \mathbb{R}^n : \|x\| < r\}$ the hypersphere of radius r.

Theorem 13 *For any $p \geq 2$, there exists $r_p > 0$ such that for any $x \in \overline{B(r_p)} \setminus \{0\}$ one has:*

(1) $V_p(x) > 0$
(2) $\langle \nabla V_p(x), f(x) \rangle < 0$

PROOF. First, we will prove that for $p = 2$, the function V_2 has the properties 1. and 2. For this, write the function f as:

$$f(x) = Ax + g(x) \qquad \text{with } A = \frac{\partial f}{\partial x}(0) \tag{18}$$

and the equation

$$\langle \nabla V(x), f(x) \rangle = -\|x\|^2 \tag{19}$$

as

$$\langle \nabla V_2(x), Ax \rangle + \langle \nabla (V - V_2)(x), Ax + g(x) \rangle + \langle \nabla V_2(x), g(x) \rangle = -\|x\|^2 \tag{20}$$

Equating the terms of second degree, we obtain:

$$\langle \nabla V_2(x), Ax \rangle = -\|x\|^2 \tag{21}$$

As $V_2(0) = 0$, it results that:

$$V_2(x) = \int\limits_0^\infty \|e^{At}x\|^2 dt \tag{22}$$

This shows that $V_2(x) > 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

On the other hand, one has:

$$\begin{aligned} \langle \nabla V_2(x), f(x) \rangle &= \langle \nabla V_2(x), Ax \rangle + \langle \nabla V_2(x), g(x) \rangle = \\ &= -\|x\|^2 + \langle \nabla V_2(x), g(x) \rangle = \\ &= -\|x\|^2 [1 - \frac{\langle \nabla V_2(x), g(x) \rangle}{\|x\|^2}] \end{aligned} \tag{23}$$

As $\lim\limits_{\|x\| \to 0} \frac{\langle \nabla V_2(x), g(x) \rangle}{\|x\|^2} = 0$, there exists $r_2 > 0$ such that for any $x \in \overline{B(r_2)} \setminus \{0\}$, we have $|\frac{\langle \nabla V_2(x), g(x) \rangle}{\|x\|^2}| < \frac{1}{2}$. Therefore, for any $x \in \overline{B(r_2)} \setminus \{0\}$, we get that:

$$\langle \nabla V_2(x), f(x) \rangle \leq -\frac{1}{2}\|x\|^2 \tag{24}$$

We will show that for any $p > 2$, the function V_p satisfies conditions 1. and 2. Write the function V_p as

$$V_p(x) = V_2(x)[1 + \frac{V_p(x) - V_2(x)}{V_2(x)}] \qquad x \neq 0 \tag{25}$$

As $\lim_{\|x\|\to 0} \frac{V_p(x)-V_2(x)}{V_2(x)} = 0$, there exists r_p^1 such that for any $x \in \overline{B(r_p^1)} \setminus \{0\}$, we have $|\frac{V_p(x)-V_2(x)}{V_2(x)}| < \frac{1}{2}$. Therefore, for any $x \in \overline{B(r_p^1)} \setminus \{0\}$, we have:

$$V_p(x) \geq \frac{1}{2} V_2(x) > 0 \tag{26}$$

thus, V_p satisfies condition 1.

On the other hand, we have:

$$\begin{aligned} \langle \nabla V_p(x), f(x) \rangle &= \langle \nabla V_2(x), Ax \rangle [1 + \frac{\langle \nabla (V_p - V_2)(x), f(x) \rangle + \langle \nabla V_2(x), g(x) \rangle}{\langle \nabla V_2(x), Ax \rangle}] = \\ &= -\|x\|^2 [1 - \frac{\langle \nabla (V_p - V_2)(x), f(x) \rangle + \langle \nabla V_2(x), g(x) \rangle}{\|x\|^2}] \end{aligned} \tag{27}$$

As $\lim_{\|x\|\to 0} \frac{\langle \nabla (V_p - V_2)(x), f(x) \rangle + \langle \nabla V_2(x), g(x) \rangle}{\|x\|^2} = 0$, there exists r_p^2 such that for any $x \in \overline{B(r_p^2)} \setminus \{0\}$, we have $|\frac{\langle \nabla (V_p - V_2)(x), f(x) \rangle + \langle \nabla V_2(x), g(x) \rangle}{\|x\|^2}| < \frac{1}{2}$. Therefore, for any $x \in \overline{B(r_p^2)} \setminus \{0\}$, we have:

$$\langle \nabla V_p(x), f(x) \rangle \leq -\frac{1}{2} \|x\|^2 \tag{28}$$

Therefore, for any $x \in \overline{B(r_p)} \setminus \{0\}$, where $r_p = \min\{r_p^1, r_p^2\}$, the function V_p satisfies conditions 1. and 2.

Corollary 14 *For any $p \geq 2$, there exists a maximal domain $G_p \subset \mathbb{R}^n$ such that $0 \in G_p$ and for any $x \in G_p \setminus \{0\}$, function V_p verifies 1. and 2. from Theorem 13. In other words, for any $p \geq 2$ the function V_p is a Lyapunov function for (1) (in the sense of [1]).*

Remark 15 *Theorem 13 provides that the Taylor polynomials of degree $p \geq 2$ associated to V in 0 are Lyapunov functions. This sequence of Lyapunov functions is different of that provided by Vanelli and Vidyasagar in [7].*

Theorem 16 *For any $p \geq 2$, there exists $c > 0$ and a closed and connected set S of points from $x \in \mathbb{R}^n$, with the following properties:*

(1) $0 \in Int(S)$
(2) $V_p(x) < c$ *for any* $x \in Int(S)$
(3) $V_p(x) = c$ *for any* $x \in \partial S$
(4) S *is compact and included in the set* G_p.

PROOF. Let be $p \geq 2$ and $r_p > 0$ determined in Theorem 13. Let be $c = \min_{\|x\|=r_p} V_p(x)$ and $S' = \{x \in \overline{B(r_p)} : V_p(x) < c\}$. It is obvious that $c > 0$ and that there exist $x^\star$ with $\|x^\star\| = r_p$ such that $V(x^\star) = c$. The set S' is open, $0 \in S'$ and $S' \subset \overline{B(r_p)} \subset G_p$.

We will prove that $V_p(x) = c$ for any $x \in \partial S'$. Let be $\bar{x} \in \partial S'$. Thus, $\|\bar{x}\| \leq r_p$ and there exists a sequence $x^k \in S'$ such that $x^k \to \bar{x}$ as $k \to \infty$. As $V_p(x^k) < c$, we have that $V_p(\bar{x}) = \lim_{k\to\infty} V_p(x^k) \leq c$. The case $\|\bar{x}\| = r_p$ and $V_p(\bar{x}) < c$ is impossible, because $c = \min_{\|x\|=r_p} V_p(x)$. The case $\|\bar{x}\| < r_p$ and $V_p(\bar{x}) < c$ is also impossible, because this would mean that $\bar{x}$ belongs to the interior of the set S', and not to its boundary. Therefore, for any $\bar{x} \in \partial S'$ we have $V_p(\bar{x}) = c$.

If the set S' is not connected (see Example 29 in this paper), we denote by S'' its connected component which contains the origin, and let be $S = \overline{S''}$. Then it is obvious that S is connected (being the closure of the open connected set S''), $0 \in Int(S) = S''$, and that for any $x \in Int(S) = S''$, we have $V_p(x) < c$. More, as $\partial S = \partial S''$, we have $V_p(x) = c$ for any $x \in \partial S$. As S'' is bounded, we obtain that the closed set S is also bounded, thus, it is compact. As $S'' \subset \overline{B(r_p)} \subset G_p$, we have that $S = \overline{S''} \subset \overline{B(r_p)} \subset G_p$. Therefore, S satisfies the properties 1-4.

Lemma 17 *Let be $p \geq 2$, $c > 0$ and a closed and connected set S satisfying 1-4 from Theorem 16. Then for any $x^0 \in S$, the solution $x(t; 0, x^0)$ of system (1) starting from x^0 is defined on $[0, \infty)$ and belongs to $Int(S)$ for any $t > 0$.*

PROOF. Let be $x^0 \in S$. We denote by $[0, \beta_{x^0})$ the right maximal interval of existence of the solution $x(t; 0, x^0)$ of system (1) with starting state x^0.

First, if $x^0 \in Int(S)\setminus\{0\}$, we show that $x(t; 0, x^0) \in Int(S)$, for all $t \in [0, \beta_{x^0})$. Suppose the contrary, i.e. there exists $T \in (0, \beta_{x^0})$ such that $x(t; 0, x^0) \in Int(S)$, for $t \in [0, T)$ and $x(T; 0, x^0) \in \partial S$ (i.e. $V_p(x(T; 0, x^0)) = c$). As $x(t; 0, x^0) \in G_p \setminus \{0\}$, for $t \in [0, T)$, $V_p(x(t; 0, x^0))$ is strictly decreasing, and it follows that $V_p(x(t; 0, x^0)) < V_p(x^0) < c$, for $t \in (0, T)$. Therefore $V_p(x(T; 0, x^0)) < c$, which contradicts the supposition $x(T; 0, x^0) \in \partial S$. Thus, $x(t; 0, x^0) \in Int(S)$, for all $t \in [0, \beta_{x^0})$. (It is clear that for $x^0 = 0$, the solution $x(t; 0, 0) = 0 \in Int(S)$, for all $t \geq 0$.)

If $x^0 \in \partial S$, we show that $x(t; 0, x^0) \in Int(S)$, for all $t \in (0, \beta_{x^0})$. As the compact set S is a subset of the domain G_p, the continuity of $x(t; 0, x^0)$ provides the existence of $T_{x^0} > 0$ such that $x(t; 0, x^0) \in G_p \setminus \{0\}$ for any $t \in [0, T_{x^0}] \subset [0, \beta_{x^0})$. Therefore $V_p(x(t; 0, x^0))$ is strictly decreasing on $[0, T_{x^0}]$, and it follows that $V_p(x(t; 0, x^0)) < V_p(x^0) = c$, for any $t \in (0, T_{x^0})$. This means that $V_p(x(t; 0, x^0)) \in Int(S)$, for any $t \in (0, T_{x^0}]$. The first part of the proof guarantees that $x(t; 0, x^0) \in Int(S)$, for all $t \in [T_{x^0}, \beta_{x^0})$, therefore, for

all $t \in (0, \beta_{x^0})$.

In conclusion, for any $x^0 \in S$, we have that $x(t; 0, x^0) \in Int(S)$, for all $t \in (0, \beta_{x^0})$.

As for any $x^0 \in S$, the solution $x(t; 0, x^0)$ defined on $[0, \beta_{x^0})$, belongs to the compact S, we obtain that $\beta_{x^0} = \infty$ and the solution $x(t; 0, x^0)$ is defined on $[0, \infty)$, for each $x^0 \in S$. More, $x(t; 0, x^0) \in Int(S)$, for all $t > 0$.

Remark 18 *Lemma 17 states that a closed and connected set S satisfying 1-4 from Theorem 16 is positively invariant to the flow of system (1).*

Theorem 19 *(LaSalle-type theorem) Let be $p \geq 2$, $c > 0$ and a closed and connected set S satisfying 1-4 from Theorem 16. Then S is a part of the of the domain of attraction $D_a(0)$.*

PROOF. Let be $x^0 \in S \setminus \{0\}$. To prove that $\lim\limits_{t\to\infty} x(t; 0, x^0) = 0$, it is sufficient to prove that $\lim\limits_{k\to\infty} x(t_k; 0, x^0) = 0$, for any sequence $t_k \to \infty$.

Consider $t_k \to \infty$. The terms of the sequence $x(t_k; 0, x^0)$ belong to the compact S. Thus, there exits a convergent subsequence $x(t_{k_j}; 0, x^0) \to y^0 \in S$.

It can be shown that

$$V_p(x(t; 0, x^0)) \geq V_p(y^0) \text{ for all } t \geq 0 \tag{29}$$

For this, observe that $x(t_{k_j}; 0, x^0) \to y^0$ and V_p is strictly decreasing along the trajectories, which implies that $V_p(x(t_{k_j}; 0, x^0)) \geq V_p(y^0)$ for any k_j. On the other hand, for any $t \geq 0$, there exists k_j such that $t_{k_j} \geq t$, and therefore $V_p(x(t; 0, x^0)) \geq V_p(x(t_{k_j}; 0, x^0)) \geq V_p(y^0)$.

We show now that $y^0 = 0$. Suppose the contrary, i.e. $y^0 \neq 0$. Inequality (29) becomes

$$V_p(x(t; 0, x^0)) \geq V_p(y^0) > 0 \text{ for all } t \geq 0 \tag{30}$$

As $V_p(x(s; 0, y^0))$ is strictly decreasing on $[0, \infty)$, we find that

$$V_p(x(s; 0, y^0)) < V_p(y^0) \text{ for all } s > 0 \tag{31}$$

For $\bar{s} > 0$, there exists a neighborhood $U_{x(\bar{s};0,y^0)} \subset S$ of $x(\bar{s}; 0, y^0)$ such that for any $x \in U_{x(\bar{s};0,y^0)}$ we have $0 < V_p(x) < V_p(y^0)$. On the other hand, for the

neighborhood $U_{x(\bar{s};0,y^0)}$ there exists a neighborhood $U_{y^0} \subset S$ of y^0 such that $x(\bar{s};0,y) \in U_{x(\bar{s};0,y^0)}$ for any $y \in U_{y^0}$. Therefore:

$$V_p(x(\bar{s};0,y)) < V_p(y^0) \text{ for all } y \in U_{y^0} \tag{32}$$

As $x(t_{k_j};0,x^0) \to y^0$, there exists $k_{\bar{j}}$ such that $x(t_{k_j};0,x^0) \in U_{y^0}$, for any $k_j \geq k_{\bar{j}}$. Making $y = x(t_{k_j};0,x^0)$ in (32), it results that

$$V_p(x(\bar{s}+t_{k_{\bar{j}}};0,x^0)) = V_p(x(\bar{s};0,x(t_{k_{\bar{j}}};0,x^0))) < V_p(y^0) \qquad \text{for } k_j \geq k_{\bar{j}} \tag{33}$$

which contradicts (30). This means that $y^0 = 0$, consequently, every convergent subsequence of $x(t_k;0,x^0)$ converges to 0. This provides that the sequence $x(t_k;0,x^0)$ is convergent to 0, for any $t_k \to \infty$, thus $\lim\limits_{t\to\infty} x(t;0,x^0) = 0$, and $x^0 \in D_a(0)$.

Therefore, the set S is contained in the domain of attraction of $D_a(0)$.

Corollary 20 *For any $p \geq 2$ and $c > 0$ there exists at most one closed and connected set satisfying 1-4 from Theorem 16.*

PROOF. Suppose the contrary, i.e. for a $p \geq 2$ and $c > 0$ there exist two different closed and connected sets S_1 and S_2 satisfying 1-4 from Theorem 16. Assume for example that there exists $x^0 \in S_1 \setminus S_2$. Due to Theorem 19, $S_1 \subset D_a(0)$ and therefore $\lim\limits_{t\to\infty} x(t;0,x^0) = 0$. As $x^0 \notin S_2$, and S_2 is a closed and connected neighborhood of 0, there exists $T > 0$ such that $x(T;0,x^0) \in \partial S_2$. Therefore, $V_p(x(T;0,x^0)) = c$ which contradicts Lemma 17. Consequently, we have $S_1 \subseteq S_2$. By the same reasons, $S_2 \subseteq S_1$. Finally, $S_1 = S_2$.

Remark 21 *If for $p \geq 2$ and $c > 0$ there exists a closed and connected set satisfying 1-4 from Theorem 16, then it is unique and it will be denoted by N_p^c.*

Corollary 22 *Any set N_p^c is included in the domain of attraction $D_a(0)$.*

Lemma 23 *Let be $p \geq 2$ and $c > 0$ such that there exists the set N_p^c. Then, for any $c' \in (0,c]$ the set $\{x \in N_p^c : V_p(x) \leq c'\}$ coincides with the set $N_p^{c'}$.*

PROOF. Let be $c' \in (0,c]$. It is obvious that $N_p^{c'}$ is included in the set $\{x \in N_p^c : V_p(x) \leq c'\}$. Let be $x^0 \in N_p^c$ such that $V_p(x^0) \leq c'$. We know that $V_p(x(t;0,x^0)) < V_p(x^0) \leq c'$, for any $t > 0$. Theorem 19 provides that $x^0 \in N_p^c \subset D_a(0)$, therefore, x^0 is connected to 0 through the continuous trajectory $x(t;0,x^0)$, along which V_p takes values below c'. In conclusion, $x^0 \in N_p^{c'}$.

Theorem 24 *If for $p \geq 2$ and $c > 0$ there exists N_p^c, then for any $c' \in (0, c)$ there exists $N_p^{c'}$ and $N_p^{c'} \subset N_p^c$. More, for any $c_1, c_2 \in (0, c)$ we have $N_p^{c_1} \subset N_p^{c_2}$ if and only if $c_1 < c_2$.*

PROOF. Lemma 23 provides that for any $c' \in (0, c)$ there exists $N_p^{c'} = \{x \in N_p^c : V_p(x) \leq c'\}$. It is obvious that $N_p^{c'} \subset N_p^c$.

Let's show that for any $c_1, c_2 \in (0, c)$ we have $N_p^{c_1} \subset N_p^{c_2}$ if and only if $c_1 < c_2$.

To show the *necessity*, let's suppose the contrary, i.e. $N_p^{c_1} \subset N_p^{c_2}$ and $c_1 \geq c_2$. Let be $x^0 \in \partial N_p^{c_2} \subset G_p$. Then $V_p(x^0) = c_2$ and as $x^0 \in G_p$, we get that

$$V_p(x(t; 0, x^0)) \leq V_p(x^0) = c_2 \qquad \text{for any } t \geq 0 \tag{34}$$

Theorem 19 provides that $x^0 \in \partial N_p^{c_2} \subset D_a(0)$, therefore $\lim\limits_{t\to\infty} x(t; 0, x^0) = 0$. As $N_p^{c_1}$ and $N_p^{c_2}$ are connected neighborhoods of 0 and $N_p^{c_1} \subset N_p^{c_2}$, there exists $T \geq 0$ such that $x(T; 0, x^0) \in \partial N_p^{c_1}$. This means that $V_p(x(T; 0, x^0)) = c_1 \geq c_2$, and (34) provides that $c_1 = c_2$. As $N_p^{c_1}$ is strictly included in $N_p^{c_2}$, there exists $\bar{x} \in \partial N_p^{c_1}$ (i.e. $V_p(\bar{x}) = c_1 = c_2$) such that $\bar{x} \in Int(N_p^{c_2})$. This contradicts the property 2 from Theorem 16 concerning $N_p^{c_2}$. In conclusion, $c_1 < c_2$.

To prove the *sufficiency*, let's suppose that $c_1 < c_2$ and let be $x^0 \in N_p^{c_1} \setminus \{0\}$. As $x^0 \in N_p^{c_1} \subset D_a(0)$, we have that $\lim\limits_{t\to\infty} x(t; 0, x^0) = 0$, so x^0 is connected to 0 through the continuous trajectory $x(t; 0, x^0)$. More, as $x^0 \in N_p^{c_1} \setminus \{0\}$, we have $V_p(x^0) \leq c_1 \leq c_2$. This means that $x^0 \in N_p^{c_2}$, therefore $N_p^{c_1} \subseteq N_p^{c_2}$. The inclusion is strict, because $N_p^{c_1} = N_p^{c_2}$ means $\partial N_p^{c_1} = \partial N_p^{c_2}$, i.e. $c_1 = c_2$, which contradicts $c_1 < c_2$.

Corollary 25 *For a given $p \geq 2$, the set of all N_p^c-s is totally ordered and $\bigcup\limits_c N_p^c$ is included in $D_a(0)$. Therefore, for a given $p \geq 2$, the largest part of $D_a(0)$ which can be found by this method is $\bigcup\limits_c N_p^c$.*

For any $p \geq 2$ let be $R_p = \{r > 0 : \overline{B(r)} \subset G_p\}$. For $r \in R_p$ we denote by $c_p^r = \inf\limits_{\|x\|=r} V_p(x)$.

Corollary 26 *For any $r \in R_p$, there exists the set $N_p^{c_p^r}$ and $N_p^{c_p^r} \subseteq \overline{B(r)}$.*

Corollary 27 *For any $p \geq 2$ and any $r', r'' \in R_p$, $r' < r''$, such that V_p is radially increasing on $\overline{B(r'')}$ we have $c_p^{r'} < c_p^{r''}$.*

Remark 28 *In some cases, it can be shown that the function V_p is radially increasing on G_p:*

a. V_2 is radially increasing on $\mathbb{R}^n$;
b. If $n = 1$, then for any $p \geq 2$, V_p is radially increasing on G_p.

This result is not true in general, provided by the following example:

Example 29 Let be the following system of differential equations:

$$\begin{cases} \dot{x_1} = -x_1 - x_1x_2 \\ \dot{x_2} = -x_2 + x_1x_2 \end{cases} \tag{35}$$

for which $(0,0)$ is an asymptotically stable steady state. For $p = 3$ the Lyapunov function $V_3(x_1, x_2)$ is given by:

$$V_3(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{3}(x_1x_2^2 - x_2x_1^2) \tag{36}$$

Consider the point $(3\sqrt{5}, \sqrt{5}) \in \partial G_3$ and let be $g : [0, 1) \to G_3$ defined by $g(\lambda) = V_3(3\sqrt{5}\lambda, \sqrt{5}\lambda)$. The function g is increasing on $[0, \frac{\sqrt{5}}{3}]$ and decreasing on $(\frac{\sqrt{5}}{3}, 1)$, therefore, the Lyapunov function V_3 is not radially increasing on the direction $(3\sqrt{5}, \sqrt{5})$. In conclusion, V_3 is not radially increasing on G_3.

More, for $c = 0.32$ there exists N_3^c, but the set $\{x = (x_1, x_2) \in G_3 : V_3(x_1, x_2) \leq c\}$ is not connected. The reason is that the point $(\bar{x_1}, \bar{x_2}) = (\frac{123}{8}, \frac{41}{24}) \in \partial G_3$ with $V_3(\bar{x_1}, \bar{x_2}) = 0$ has a nonempty neighborhood U such that $V_3(x_1, x_2) \leq c$, for any $(x_1, x_2) \in G_3 \bigcap U$ and $(G_3 \bigcap U) \bigcap N_3^c = \emptyset$.

Theorem 30 *For any $p \geq 2$ there exists $\rho_p > 0$ such that V_p is radially increasing on $\overline{B(\rho_p)}$.*

PROOF. It can be easily verified that V_2 is radially increasing on $\mathbb{R}^n$, using relation (22). This provides that for any $x \in \mathbb{R}^n \setminus \{0\}$, the function $g_2^x : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $g_2^x(\lambda) = V_2(\lambda x)$ is strictly increasing on $\mathbb{R}_+$, therefore $\frac{d}{d\lambda} g_2^x(\lambda) > 0$ on $\mathbb{R}_+^\star$, i.e.

$$\langle \nabla V_2(\lambda x), x \rangle > 0 \qquad \text{for any } \lambda > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\} \tag{37}$$

Let be $p > 2$, $x \in \mathbb{R}^n \setminus \{0\}$ and $g_p^x : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $g_p^x(\lambda) = V_p(\lambda x)$. One has:

$$\frac{d}{d\lambda} g_p^x(\lambda) = \langle \nabla V_p(\lambda x), x \rangle = \langle \nabla V_2(\lambda x), x \rangle + \langle \nabla (V_p - V_2)(\lambda x), x \rangle =$$

$$= \langle \nabla V_2(\lambda x), x\rangle (1 + \frac{\langle \nabla (V_p - V_2)(\lambda x), x\rangle}{\langle \nabla V_2(\lambda x), x\rangle}) \tag{38}$$

As $\lim\limits_{x\to 0} \frac{\langle \nabla (V_p - V_2)(\lambda x), x\rangle}{\langle \nabla V_2(\lambda x), x\rangle} = 0$, there exists $\rho_p > 0$ such that $|\frac{\langle \nabla (V_p - V_2)(\lambda x), x\rangle}{\langle \nabla V_2(\lambda x), x\rangle}| \leq \frac{1}{2}$, for any $x \in \overline{B(\rho_p)} \setminus \{0\}$. Relation (38) provides that for any $x \in \overline{B(\rho_p)} \setminus \{0\}$, we have:

$$\frac{d}{d\lambda} g_p^x(\lambda) \geq \frac{1}{2} \langle \nabla V_2(\lambda x), x\rangle > 0 \qquad \text{for any } \lambda > 0 \tag{39}$$

Therefore, for any $x \in \overline{B(\rho_p)} \setminus \{0\}$, the function g_p^x is strictly increasing on $\mathbb{R}^n$, i.e. V_p is radially increasing on $\overline{B(\rho_p)}$.

Theorem 31 *Let be $p \geq 2$ and $c > 0$ such that there exists the set N_p^c. Suppose that for any $c' \leq c$, the sets $N_p^{c'}$ have the star-property, i.e. for any $x \in N_p^{c'}$ and for any $\lambda \in [0, 1)$ one has $\lambda x \in Int(N_p^{c'})$. Then V_p is radially increasing on N_p^c.*

PROOF. Let be $x^0 \in \partial N_p^c$ and $0 < \lambda_1 < \lambda_2 \leq 1$. We have to show that $V_p(\lambda_1 x^0) < V_p(\lambda_2 x^0)$. Denote $c_1 = V_p(\lambda_1 x^0) > 0$, $c_2 = V_p(\lambda_2 x^0) > 0$ and suppose the contrary, i.e. $c_1 \geq c_2$. Theorem 24 provides that $N_p^{c_1} \supseteq N_p^{c_2}$. Lemma 23 guarantees that $\lambda_2 x \in \partial N_p^{c_2}$. As $N_p^{c_2}$ has the star-property, then for $\lambda = \frac{\lambda_1}{\lambda_2} \in (0, 1)$, we have that $\lambda(\lambda_2 x) = \lambda_1 x \in Int(N_p^{c_2})$, so $c_1 = V_p(\lambda_1 x) < c_2$ which contradicts the supposition $c_1 \geq c_2$. Therefore, V_p is radially increasing on N_p^c.

Remark 32 *a. For any $x \in D_0$, there exists $p_x \geq 2$ such that $x \in G_p$ for any $p \geq p_x$;*

b. If $n = 1$, there exists $p_0 \geq 2$ such that $D_0 \subset G_p$, for any $p \geq p_0$.

c. If there exists $r > 0$ such that $\overline{B(r)} \subset G_p$ for any $p \geq 2$, then there exists $p_0 \geq 2$ such that $D_0 \subset G_{p_0}$.

Conjecture 33 *For any $x \in D_a(0)$ there exists $p \geq 2$ and $c > 0$ such that $x \in N_p^c$.*

3 Numerical example: the Van der Pol system

We consider the following system of differential equations:

$$\begin{cases} \dot{x_1} = -x_2 \\ \dot{x_2} = x_1 - x_2 + x_1^2 x_2 \end{cases} \tag{40}$$

The $(0,0)$ steady state of (40) is asymptotically stable. The boundary of the domain of attraction of $(0,0)$ is a limit cycle of (40).

For $p = 20$ we have computed that the largest value $c > 0$ for which there exists the set N_p^c is $c_{20} = 8.8466$. For $p = 50$, the largest value $c > 0$ for which there exists the set N_p^c is $c_{50} = 13.887$. In the figures below, the thick black curve represents the boundary of $D_a(0,0)$, the thin black curve represents the boundary of G_p and the gray surface represents the set $N_p^{c_p}$. The set $N_p^{c_{50}}$ approximates very well the domain of attraction of $(0,0)$.

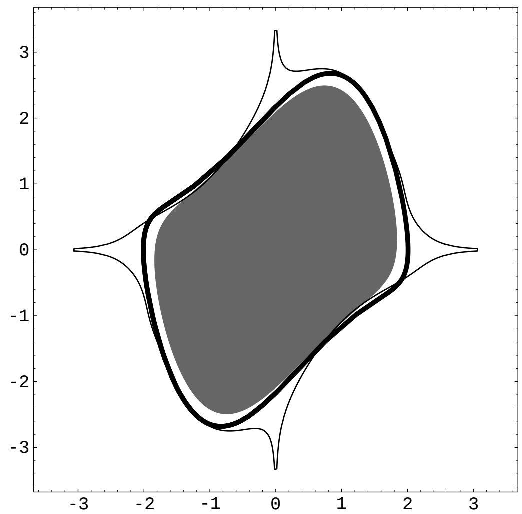

Fig. 1. The sets $N_p^{c_{20}}$, G_{20} and $D_a(0,0)$ for system (40)

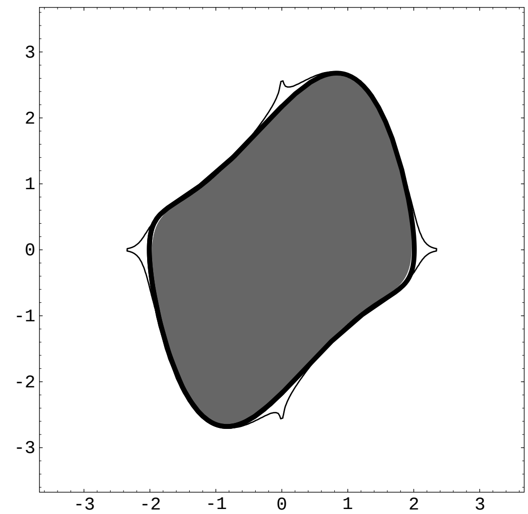

Fig. 2. The sets $N_p^{c_{50}}$, G_{50} and $D_a(0,0)$ for system (40)

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